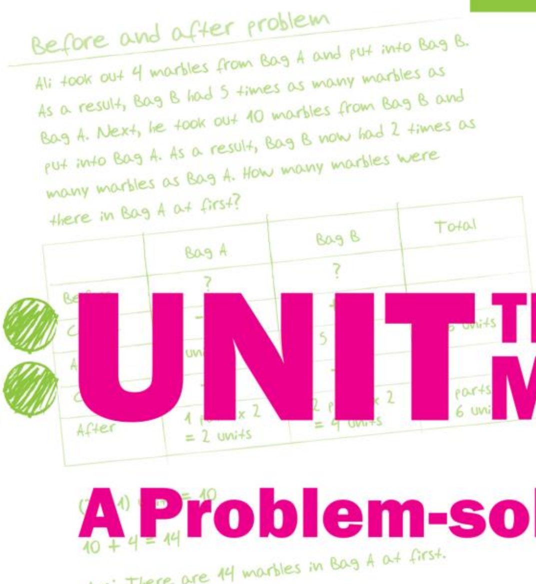
Mastering Heuristics Series
Handbooks for discerning parents



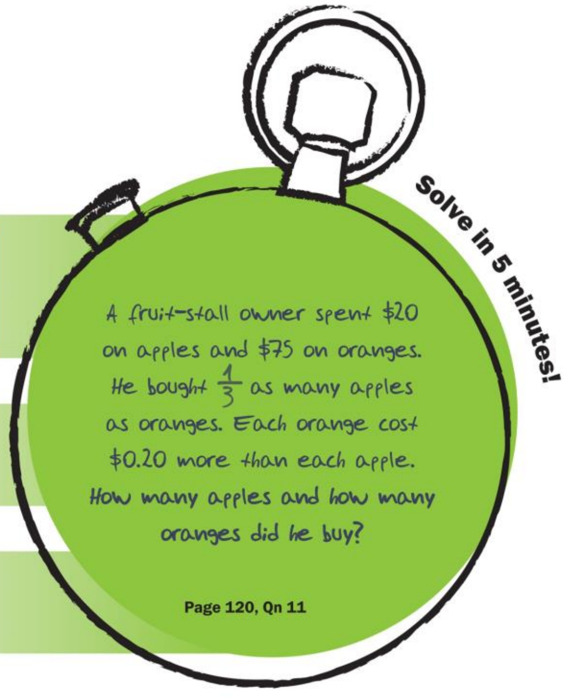
1 UNIT TRANSFER METHOD

A Problem-solving Tool

for Challenging Problems in Upper Primary Mathematics

Sunny Tan

Primary 4



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Unit Transfer Method
A Problem-solving Tool
for Challenging Problems
in Upper Primary Mathematics
(Primary 4)

Sunny Tan
Maths Heuristics (S) Private Limited

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PREFACE

This book teaches Unit Transfer Method at Primary 4, which is the use of proportion to effectively analyse and solve challenging mathematical problems involving whole numbers, fractions and decimals. It delivers the foundation for learning Unit Transfer Method at Primary 5 where mathematical problems are expanded to involve ratios and percentages.

Ultimately, Unit Transfer Method is a simple, logical yet powerful problem-solving technique that complements the model approach and the algebraic approach. It also paves the way for students to eventually learn the algebraic approach in secondary school.

Heuristics in Primary Maths Syllabus

Heuristics is a specialised mathematical problem-solving concept. Mastering it facilitates efficiency in solving regular as well as challenging mathematical problems. The Ministry of Education in Singapore has incorporated 11 Problem-Solving Heuristics into all primary-level mathematical syllabus.

Learning Heuristics Effectively

Instead of containing the 11 Problem-Solving Heuristics neatly into specific chapters though, they have been integrated into the regular curriculum. This not only makes it difficult for students to pick up Heuristics skills, but can also make mathematics confusing for some students. For us parents, it is difficult for us to put aside the regular-syllabus mathematical concepts we were brought up on to re-learn Heuristics, much less teach our own children this new concept.

Take Algebraic Equations, for instance. Although it is a Heuristics technique, the topic has never been, and is still not, taught at primary level. Yet, primary-level Mathematics Papers these days include questions from the topic. Parents, being familiar with the topic, will attempt to teach their children to solve the questions using Algebraic Equations. This will only confuse their children. According to current primary-level mathematical syllabus, other Heuristics techniques should be used instead.

These and other challenges were what I observed first-hand during my years as a mathematics teacher, and what provided me the impetus for my post-graduate studies, mathsHeuristics™ programmes and, now, the Mastering Heuristics Series of books.

mathsHeuristics™ programmes and the Mastering Heuristics Series of books cater to different levels, from Primary 4 to Primary 6. Students with a firm foundation in heuristics at Primary 4 will experience a smoother transition to Primary 5 and Primary 6 mathematics. The Primary 5 and Primary 6 mathematical syllabi no longer aim to teach heuristics techniques, focusing instead on technique application in problem-solving situations. As such, it is important for Primary 4 students to become familiar and comfortable with heuristics techniques before they move up to Primary 5.

About Mastering Heuristics Series

This series of books is a culmination of my systematic thinking, supported by professional instructional writing and editing, to facilitate understanding and mastery of Heuristics. I would also like to acknowledge the support from two individuals in local academia – Mr Ammiel Wan, an authority in local mathematics syllabus, who shared his knowledge in and experience with the teaching of heuristics. Dr Lua Eng Keong, an overseas academic and a Singapore parent, who is now back in Singapore. Through his then 9-year-old son, feedback and suggestions, Dr Lua provided a test bed and refinement opportunities for the methodologies used to teach heuristics in the Mastering Heuristics Series.

Through this series, I have neatly packaged Heuristics into different techniques (Series of books) and application scenarios (Chapters within each book). For each application scenario, I offer many examples, showing how the technique may be applied, and then explaining the application in easy-to-follow steps and visualisations without skipping a beat.

The entire series of books provides a complete and comprehensive guide to Heuristics.

Multi Benefits to Students, Parents and Educators

While each book introduces parents to one Heuristics technique applied across diverse scenarios, it gives students the opportunity to see how the specific Heuristics technique works as well as get in some practice.

For students enrolled in mathsHeuristics™ programmes, each book serves as a great companion, while keeping parents well-informed of what their children are learning.

Sunny Tan

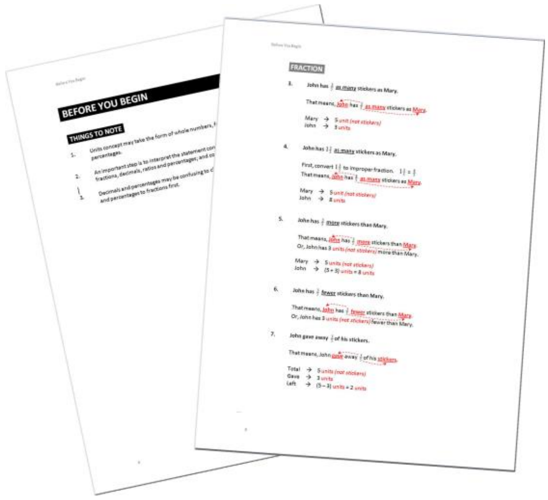
August 2011

HOW TO USE THIS BOOK

BEFORE YOU BEGIN

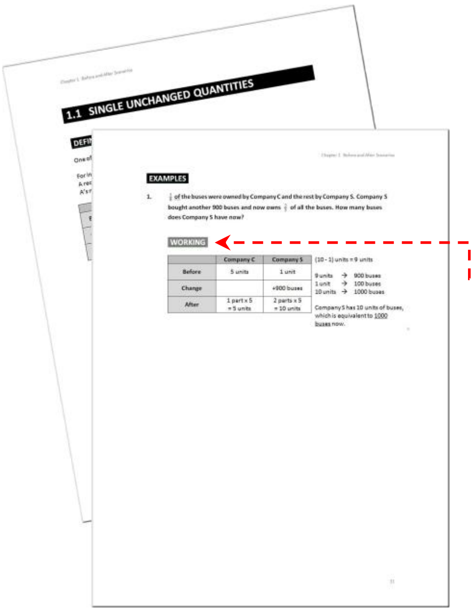
The “Before You Begin” chapter instills the basic but important step that must be applied across every question under this topic. This helps to standardise the given information for easy application of the techniques being taught.

In this book on Unit Transfer Method (UTM) at Primary 4, this step is to convert whole numbers, fractions and decimals into units.



CHAPTERS AND SECTIONS

The various UTM techniques are neatly separated into different chapters and sections. Thus, examples of UTM application are classified according to problem-solving techniques for more focused learning.

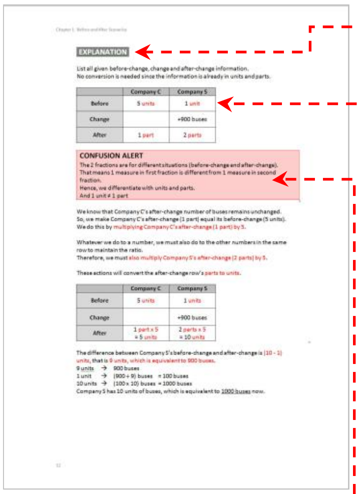


EXAMPLES

Each example of UTM application comes with “Working” and “Explanation”, which includes “Confusion Alert” boxes.

WORKING

“Working” shows heuristics application in action (how quick it is to solve a question).



EXPLANATION

"Explanation" shows the thought process behind the heuristics application (the detailed steps). It takes readers through the solution in the following manner:

- step-by-step without skipping a beat so that readers can follow what happens at each and every step.
- systematic so that readers begin to see a pattern in applying the technique.
- easy-to-follow so that readers can quickly understand the technique minus the frustration.

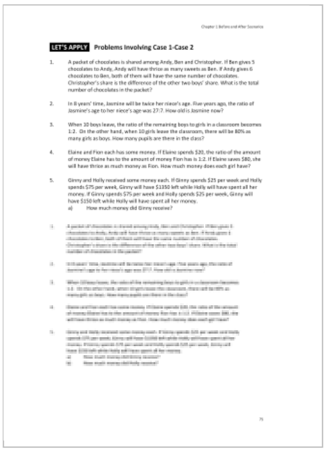
In UTM, readers will see that its application always begins with:

- the basic step explained in "Before You Begin", and
- the tabulation of all given information.

This quickly helps students see and understand the relationships among all the information given in the question.

"Confusion Alert" boxes in the midst of "Explanation" highlight areas where students are likely to be uncertain of or make mistakes in. It also gives the rationale to help clarify doubts in these areas.

LET'S APPLY



Learning is only effective when followed up with practice. Hence, at the end of each chapter/section is a list of questions related to the heuristics technique taught in that chapter/section.

ADDITIONAL TIPS

For on-going sharing and discussions on the use of UTM, visit:
www.unittransfermethod.blogspot.com

For detailed workings to all UTM "Let's Apply" section, visit:
www.mathsHeuristics.com/?page_id=472

BEFORE YOU BEGIN

THINGS TO NOTE

1. The units concept may take the form of whole numbers, fractions and decimals.

2. An important step is to interpret the statement containing whole numbers, fractions and decimals; and convert them into units.

3. Decimals may be confusing to children. So, convert any decimals to fractions first.

EXAMPLES

WHOLE NUMBER

1. **John has 5 times <u>as many</u> stickers as Mary.**

First, convert the whole number to fraction. $5 = \frac{5}{1}$

That means, <u>John</u> has $\frac{5}{1}$ times <u>as many</u> stickers as <u>Mary</u>.

Mary → 1 unit *(not stickers)*
John → 5 units

2. **John has 5 times <u>more</u> stickers than Mary.**

First, convert the whole number to fraction. $5 = \frac{5}{1}$

That means, <u>John</u> has $\frac{5}{1}$ times <u>more</u> stickers than <u>Mary</u>.

Mary → 1 unit *(not stickers)*
John → (1 + 5) units = 6 units

FRACTIONS

3. **John has $\frac{3}{5}$ <u>as many</u> stickers as Mary.**

That means, <u>**John**</u> has $\frac{3}{5}$ <u>**as many**</u> stickers as <u>**Mary**</u>.

Mary → 5 units *(not stickers)*
John → 3 units

4. **John has $1\frac{3}{5}$ <u>as many</u> stickers as Mary.**

First, convert $1\frac{3}{5}$ to improper fraction. $1\frac{3}{5} = \frac{8}{5}$
That means, <u>**John**</u> has $\frac{8}{5}$ <u>**as many**</u> stickers as <u>**Mary**</u>.

Mary → 5 units *(not stickers)*
John → 8 units

5. **John has $\frac{3}{5}$ <u>more</u> stickers than Mary.**

That means, <u>**John**</u> has $\frac{3}{5}$ <u>**more**</u> stickers than <u>**Mary**</u>.
Or, John has 3 units *(not stickers)* more than Mary.

Mary → 5 units *(not stickers)*
John → (5 + 3) units = 8 units

6. **John has $\frac{3}{5}$ <u>fewer</u> stickers than Mary.**

That means, <u>**John**</u> has $\frac{3}{5}$ <u>**fewer**</u> stickers than <u>**Mary**</u>.
Or, John has 3 units *(not stickers)* fewer than Mary.

Mary → 5 units *(not stickers)*
John → (5 – 3) units = 2 units

7. **John gave away $\frac{3}{5}$ of his stickers.**

That means, John <u>**gave**</u> away $\frac{3}{5}$ of his <u>**stickers**</u>.

Total → 5 units *(not stickers)*
Gave → 3 units
Left → (5 – 3) units = 2 units

DECIMALS

8. **John has 0.6 times as many stickers as Mary.**

First, convert the decimal to fraction. 0.6 = $\frac{6}{10}$ = $\frac{3}{5}$

That means, **John** has $\frac{3}{5}$ **as many** stickers as **Mary**. **(Just like Example 3)**

Mary → 5 units *(not stickers)*
John → 3 units

9. **John has 1.6 times as many stickers as Mary.**

First, convert the decimal to improper fraction. 1.6 = $1\frac{6}{10}$ = $1\frac{3}{5}$ = $\frac{8}{5}$

That means, **John** has $\frac{8}{5}$ **as many** stickers as **Mary**. **(Just like Example 4)**

Mary → 5 units *(not stickers)*
John → 8 units

10. **John has 0.6 times more stickers than Mary.**

First, convert the decimal to fraction. 0.6 = $\frac{6}{10}$ = $\frac{3}{5}$

That means, **John** has $\frac{3}{5}$ **more** stickers than **Mary**. **(Just like Example 5)**

Or, John has 3 units *(not stickers)* more than Mary.

Mary → 5 units *(not stickers)*
John → (5 + 3) units = 8 units

11. **John has 0.6 times fewer stickers than Mary.**

First, convert the decimal to fraction. 0.6 = $\frac{6}{10}$ = $\frac{3}{5}$

That means, **John** has $\frac{3}{5}$ **fewer** stickers than **Mary**. **(Just like Example6)**

Or, John has 3 units *(not stickers)* fewer than Mary.

Mary → 5 units *(not stickers)*
John → (5 – 3) units = 2 units

12. **John gave away 0.6 of his stickers.**

First, convert the decimal to fraction. 0.6 = $\frac{6}{10}$ = $\frac{3}{5}$

That means, John **gave** away $\frac{3}{5}$ of his **stickers**. **(Just like Example 7)**

Total → 5 units *(not stickers)*
Gave → 3 units
Left → (5 – 3) units = 2 units

CHAPTER 1 BEFORE AND AFTER SCENARIOS

STEPS

- List all given before-action, change-action and after-action information.
- Convert the information into units and parts,
  where necessary and if not already in units and parts.
- Compare the information to find the unknown.

APPLICABILITY

There are four basic scenarios where the Before and After Concept may be applied.

- Single Unchanged Quantities
- Total Unchanged Quantities
- Difference Unchanged Quantities
- All Changing Quantities

1.1 SINGLE UNCHANGED QUANTITIES

DEFINITION

One of the given quantities remains unchanged.

ILLUSTRATION

For instance, A and B have stickers in a certain quantities (Say, 20 and 50 stickers).
A receives 5 stickers from C (external party).
A’s number of stickers changed, B’s number of stickers remains unchanged.

	A	B
Before	20	50
Change	+5	
After	25	50

→ In the change row, the change figure appears in the column where the change occurred.

↓ In the column where the change cell is empty (no change occurred), the before-change and after-change quantities remain the same (quantities unchanged).

While A’s number of stickers changes (+5), B’s number of stickers remains unchanged (Single Unchanged Quantity).

EXAMPLES

1. **Wilson had 4 times as many fish as Ian. After Ian bought 3 additional fish from the pet shop, Ian had 0.5 times as many fish as Wilson. How many fish did Wilson have at first?**

WORKING

	Wilson's fish	Ian's fish
Before	4 units	1 unit
Change		+3 fish
After	2 parts x 2 = 4 units	1 part x 2 = 2 units

(2 – 1) units = 1 unit

1 unit → 3 fish
4 units → (4 x 3) fish = 12 fish

Wilson had 12 fish at first.

EXPLANATION

List all given before-change, change and after-change information.
No conversion is needed since the information is already in units and parts.

	Wilson's fish	Ian's fish
Before	4 units	1 unit
Change		+ 3 fish
After	2 parts	1 part

CONFUSION ALERT
The ratio and decimal are for different situations (before-change and after-change). That means 1 measure in first whole number is different from 1 measure in second decimal. Hence, we differentiate with units and parts.
And 1 unit ≠ 1 part

We know that Wilson's after-change number of fish remains unchanged.
So, we make Wilson's after-change (2 parts) equal its before-change (4 units).
We do this by multiplying Wilson's after-change (2 parts) by 2.

Whatever we do to a number, we must also do to the other numbers in the same row to maintain the ratio.
Therefore, we must also multiply Ian's after-change (1 part) by 2.

These actions will convert the after-change row's parts to units.

	Wilson's fish	Ian's fish
Before	4 units	1 unit
Change		+3 fish
After	2 parts x 2 = 4 units	1 part x 2 = 2 units

The difference between Ian's before-change and after-change is (2 – 1) unit, that is 1 unit.
We know that Ian bought 3 additional fish from the pet shop.
1 unit → 3 fish

Wilson had 4 units of fish at first.
4 units → (4 x 3) fish = 12 fish

Wilson had 12 fish at first.

2. **The number of women at a zoo was $\frac{7}{10}$ the number of men. After 121 women left the zoo, the number of women remaining became $\frac{1}{3}$ the number of men remaining. How many women were there at the zoo at first?**

WORKING

	Men	Women
Before	10 units x 3 = 30 units	7 units x 3 = 21 units
Change		– 121 women
After	3 parts x 10 = 30 units	1 part x 10 = 10 units

(21 – 10) units = 11 units

11 units → 121 women
1 unit → (121 ÷ 11) women = 11 women
21 units → (21 x 11) women = 231 women

There were 231 women at the zoo at first.

EXPLANATION

List all given before-change, change and after-change information.
No conversion is needed since the information is already in units and parts.

	Men	Women
Before	10 units	7 units
Change		– 121 women
After	3 parts	1 part

CONFUSION ALERT
The 2 fractions are for different situations (before-change and after-change).
That means 1 measure in first fraction is different from 1 measure in second fraction. Hence, we differentiate with units and parts.
And 1 unit ≠ 1 part

We know that the number of men remains unchanged.
So, we make the men's after-change (3 parts) equal their before-change (10 units).
We do this by multiplying the men's after-change (3 parts) by 10,
and their before-change (10 units) by 3.

Whatever we do to a number, we must also do to the other numbers in the same row to maintain the ratio.
Therefore, we must also multiply the women's after-change (1 part) by 10,
and their before-change (7 units) by 3.

These actions will convert the after-change row's parts to units.

	Men	Women
Before	10 units x 3 = 30 units	7 units x 3 = 21 units
Change		– 121 women
After	3 parts x 10 = 30 units	1 part x 10 = 10 units

The difference between the women's before-change and after-change is (21 – 10) units, that is 11 units.
We know that 121 women left the zoo.
11 units → 121 women
1 unit → (121 ÷ 11) women = 11 women

There were 21 units of women at the zoo at first.
21 units → (21 x 11) men = 231 women

There were 231 women at the zoo at first.

3. **Colin had 0.7 times as many stamps as Daniel. After Daniel gave away 54 stamps to his friends, both boys had an equal number of stamps left. How many stamps did Daniel have at first?**

WORKING

$0.7 = \frac{7}{10}$

	Colin's stamps	Daniel's stamps
Before	7 units	10 units
Change		– 54 stamps
After	1 part x 7 = 7 units	1 part x 7 = 7 units

(10 – 7) units = 3 units

3 units → 54 stamps
1 unit → (54 ÷ 3) stamps = 18 stamps
10 units → (10 x 18) stamps = 180 stamps

Daniel had 180 stamps at first.

EXPLANATION

Convert the decimal to fraction first.
$0.7 = \frac{7}{10}$

List all given before-change, change and after-change information.
No conversion is needed since the information is already in units and parts.

	Colin's stamps	Daniel's stamps
Before	7 units	10 units
Change		– 54 stamps
After	1 part	1 part

CONFUSION ALERT
The decimal and ratio are for different situations (before-change and after-change). That means 1 measure in first decimal is different from 1 measure in second whole number. Hence, we differentiate with units and parts.
And 1 unit ≠ 1 part

We know that the number of Colin's stamps remains unchanged.
So, we make the Colin's stamps' after-change (1 part) equal its before-change (7 units).
We do this by multiplying Colin's after-change (1 part) by 7,
and his before-change (7 units) by 1.

Whatever we do to a number, we must also do to the other numbers in the same row to maintain the ratio.
Therefore, we must also multiply Daniel's after-change (1 part) by 7,
and his before-change (10 units) by 1.

These actions will convert the after-change row's parts to units.

	Colin's stamps	Daniel's stamps
Before	7 units x 1 = 7 units	10 units x 1 = 10 units
Change		– 54 stamps
After	1 part x 7 = 7 units	1 part x 7 = 7 units

The difference between Daniel's before-change and after-change is (10 - 7) units, that is 3 units.
We know that Daniel gave away 54 stamps.
3 units → 54 stamps
1 unit → (54 ÷ 3) stamps = 18 stamps

Daniel had 10 units of stamps at first.
10 units → (10 x 18) stamps = 180 stamps

Daniel had 180 stamps at first.

LET'S APPLY **Problems Involving Single Unchanged Quantities**

1. Tarzan had twice as many monkeys as Jane. After 120 of Tarzan's monkeys ran away, Jane had thrice as many monkeys as Tarzan. How many monkeys did Tarzan have at first?

2. Eve had twice as many apples as Adam. After Adam gave 8 of his apples to his children, Adam had $\frac{1}{3}$ as many apples as Eve. How many apples did Eve and Adam have altogether at first?

3. Lily had 3 times as many pairs of earrings as necklaces. After giving away 18 necklaces, she had 5 times as many pairs of earrings as necklaces. How many pairs of earrings did she have?

4. Daisy had 3 times as many candies as Rose. After Daisy gave 6 candies to her sister, she had 1.5 times as many candies as Rose. How many candies did Rose have?

5. $\frac{7}{13}$ of Grace's beads are red and the rest are purple. If she gives 15 red beads away, she will have an equal number of red beads and purple beads. How many purple beads does she have?

6. Benny and Tom had the same number of stamps. When Benny gave away 147 of his stamps, Benny had 0.25 times as many stamps as Tom. How many stamps did the two boys have altogether at first?

7. There were some marbles and 42 rubber balls in a box. Jessica took out 20 marbles from the box. As a result, there were twice as many rubber balls as marbles in the box. How many marbles were there in the box at first?

8. There were thrice as many children as adults in a hall. After 7 adults left the hall, there were 6 times as many children as adults in the hall. How many people were there in the hall at first?

9. Janice had 5 times as many boxes of chocolates as Jarrod. If Jarrod gave away 6 boxes of chocolates to his sister, he would have $\frac{1}{8}$ as many boxes of chocolates as Janice.
   a) How many boxes of chocolates did Janice have?
   b) If each box contained 12 pieces of chocolates, how many pieces of chocolates does Janice have in all?

10. Pauline has $\frac{1}{8}$ times as many red packets as Esther. Pauline receives 9 more red packets. Now, she has 0.5 times as many red packets as Esther. How many red packets does Esther have?

1.2 TOTAL UNCHANGED QUANTITIES

DEFINITION

The total quantities remain unchanged.

ILLUSTRATION

For instance, A and B have stickers in different quantities (Say, 20 and 50 stickers).
A gives B (both internal parties) 5 stickers.
A’s and B’s respective number of stickers changes,
but A and B’s total number of stickers remains unchanged Before-transfer and After-transfer.

	A	B	Total
Before	20	50	70
Change	-5	+5	
After	15	55	70

→ In the change row, the change figure appears in both A and B columns. While the quantities are the same, the signs are different (+/-).

↓ A simply made a transfer to B.
This does not change A and B’s total.
So, the change cell is empty (no change occurred).
And the before-change and after-change Total remain the same (Total Unchanged Quantities).

EXAMPLES

1. **Peggy had thrice as many paper stars as Mike. After Mike received 27 paper stars from Peggy, the two children had an equal number of paper stars. How many paper stars did Peggy have at first?**

WORKING

	Peggy's paper stars	Mike's paper stars	Total paper stars
Before	3 units	1 unit	(3 + 1) units = 4 units
Change	- 27 paper stars	+ 27 paper stars	
After	1 part x 2 = 2 units	1 part x 2 = 2 units	(1 + 1) parts x 2 = 4 units

(3 - 2) units = 1 unit

1 unit → 27 paper stars
3 units → (3 x 27) paper stars = 81 paper stars

Peggy had 81 paper stars at first.

EXPLANATION

List all given before-change, change and after-change information.
No conversion is needed since the information is already in units and parts.

	Peggy's paper stars	Mike's paper stars	Total paper stars
Before	3 units	1 unit	(3 + 1) units = 4 units
Change	- 27 paper stars	+ 27 paper stars	
After	1 part	1 part	(1 + 1) parts = 2 parts

CONFUSION ALERT

The 2 ratios are for different situations (before-change and after-change).
That means 1 measure in the first ratio is different from 1 measure in the second ratio. Hence, we differentiate with units and parts.
And 1 unit ≠ 1 part

We know that the total number of paper stars remains unchanged.
So, we make the total number of paper stars after-change (2 parts) equal before-change (4 units).
We do this by multiplying the total number of paper stars after-change (2 parts) by 2, and its before-change (4 units) by 1.

Whatever we do to a number, we must also do to the other numbers in the same row to maintain the ratio.
Therefore, we must multiply the after-change row (1 part and 1 part) by 2,
and the before-change row (3 units and 1 unit) by 1.

These actions will convert the after-change row's parts to units.

	Peggy's paper stars	Mike's paper stars	Total paper stars
Before	3 units x 1 = 3 units	1 unit x 1 = 1 unit	4 units x 1 = 4 units
Change	- 27 paper stars	+ 27 paper stars	
After	1 part x 2 = 2 units	1 part x 2 = 2 units	2 parts x 2 = 4 units

The change in each child's number of paper stars is (3 - 1) **OR** (2 - 1) units = 1 unit
We know Peggy gave away 27 paper stars while Mike received 27 paper stars.
1 unit → 27 paper stars

Peggy had 3 units at first.
3 units → (3 x 27) paper stars = 81 paper stars

Peggy had 81 paper stars at first.

2. **Helen had 0.5 as many sweets as Tam. After Helen gave 28 sweets to Tam, Tam had 4 times as many sweets as Helen. How many sweets did both children have altogether?**

WORKING

0.5 = $\frac{5}{10}$ = $\frac{1}{2}$

	Helen's sweets	Tam's sweets	Total sweets
Before	1 unit x 5 = 5 units	2 units x 5 = 10 units	(2 + 1) units x 5 = 15 units
Change	- 28 sweets	+ 28 sweets	
After	1 part x 3 = 3 units	4 parts x 3 = 12 units	(1 + 4) parts x 3 = 15 units

(5 - 3) units = 2 units

2 units → 28 sweets
1 unit → (28 ÷ 2) sweets = 14 sweets
15 units → (15 x 14) sweets = 210 sweets

They have 210 sweets altogether.

EXPLANATION

List all given before-change, change and after-change information.
No conversion is needed since the information is already in units and parts.

	Helen's sweets	Tam's sweets	Total sweets
Before	1 unit	2 units	(1 + 2) units = 3 units
Change	- 28 sweets	+ 28 sweets	
After	1 part	4 parts	(1 + 4) parts = 5 parts

CONFUSION ALERT
The decimal and ratio are for different situations (before-change and after-change).
That means 1 measure in the first decimal is different from 1 measure in the second ratio. Hence, we differentiate with units and parts.
And 1 unit ≠ 1 part

We know that the total sweets remain unchanged.
So, we make the total sweets after-change (5 parts) equal its before-change (3 units).
We do this by multiplying the total sweets after-change (5 parts) by 3,
and its before-change (3 units) by 5.

Whatever we do to a number, we must also do to the other numbers in the same row to maintain the ratio.
Therefore, we must multiply the after-change row (1 part and 4 parts) by 3,
and the before-change row (1 unit and 2 units) by 5.

These actions will convert the after-change row's parts to units.

	Helen's sweets	Tam's sweets	Total sweets
Before	1 unit x 5 = 5 units	2 units x 5 = 10 units	3 units x 5 = 15 units
Change	- 28 sweets	+ 28 sweets	
After	1 part x 3 = 3 units	4 parts x 3 = 12 units	5 parts x 3 = 15 units

The change in number of sweets for each child is (5 - 3) **OR** (12 - 10) units, that is 2 units.
We know that Helen gave 28 sweets to Tam.
2 units → 28 sweets
1 unit → (28 ÷ 2) sweets = 14 sweets

Helen and Tam had 15 units of sweets altogether.
15 units → (15 x 14) sweets = 210 sweets

They had 210 sweets altogether.

3. **Marcus had two boxes of marbles. After transferring $\frac{7}{19}$ of the marbles from Box B to Box A, the marbles in Box B became 1.5 times as many marbles as that in Box A. What fraction of the marbles was there in Box A at first?**

WORKING

	Marbles in Box A	Marbles in Box B	Total
Before	? units = (20 - 19) units	19 units	20 units
Change	+ 7 units	- 7 units	
After	2 parts x 4 = 8 units	3 parts x 4 = 12 units	(2 + 3) parts x 4 = 20 units

The fraction of the number of marbles in Box A at first is $\frac{1}{20}$.

EXPLANATION

List all given before-change, change and after-change information.
No conversion is needed since the information is already in units and parts.

	Marbles in Box A	Marbles in Box B	Total
Before	? units	19 units	
Change	+ 7 units	- 7 units	
After	2 parts	3 parts ↓ (19 - 7) units = 12 units	(2 + 3) parts = 5 parts

CONFUSION ALERT

The fraction and decimal are for different situations (before-change and after-change). That means 1 measure in the fraction is different from 1 measure in the decimal. Hence, we differentiate with units and parts.
And 1 unit ≠ 1 part

We know that the number of marbles in Box B after-change is (19 - 7) units, that is 12 units.
So, we make the number of marbles in Box B after-change (3 parts) equal 12 units.
We do this by multiplying the number of marbles in Box B after-change (3 parts) by 4.

Whatever we do to a number, we must also do to the other numbers in the same row to maintain the ratio.
Therefore, we must multiply the after-change row (2 parts and 5 parts) by 4.

These actions will convert the after-change row's parts to units.

	Marbles in Box A	Marbles in Box B	Total
Before	? units	19 units	
Change	+ 7 units	- 7 units	
After	2 parts x 4 = 8 units	3 parts x 4 = 12 units	5 parts x 4 = 20 units

Since the total number of marbles did not change,
the total number of marbles before-change must also be 20 units.

	Marbles in Box A	Marbles in Box B	Total
Before	? units	19 units	20 units
Change	+ 7 units	- 7 units	
After	8 units	12 units	20 units

Therefore, the number of marbles in Box A before-change is (20 – 19) units, that is 1 unit.
The fraction of marbles Box A at first is $\frac{1}{20}$.

LET'S APPLY Problems Involving Total Unchanged Quantities

1. Jenny has 156 buttons and Lin has 274 buttons. How many buttons must Lin give to Jenny so that each of them will have the same number of buttons?

2. Alden and Glenn had the same amount of savings in their piggy banks. After Alden gave $21.30 to Glenn, Glenn had 4 times as much savings as Alden. How much money did they have altogether?

3. Pat has 50 colour pencils. Melissa has 174 colour pencils. How many colour pencils must Melissa give Pat so that the number of colour pencils Pat has is 3 times as many as that of Melissa's?

4. Mary and Karen have 40 dolls. After Karen gives 2 dolls to Mary, she has thrice as many dolls as Mary. How many dolls does Karen have at first?

5. William's savings is $24 more than Nicholas's savings. If William gives $6 to Nicholas, how much more money does William have than Nicholas now?

6. Zen had 45 sweets. After giving 8 sweets to Ying, he had 4 sweets less than Ying. How many sweets did Ying have at first?

7. There were 155 stamps in Album A and 63 stamps in Album B. How many stamps must be transferred from Album A to Album B so that Album B has 12 more stamps than Album A?

8. Jerry and Tom had some sweets. After Tom gave 100 sweets to Jerry, Tom found that they had the same number of sweets. If Jerry had 80 sweets at first, how many sweets did Tom have at first?

9. David has 420 stamps which is $\frac{1}{4}$ as many as what Joyce had. How many stamps must Joyce give to David so that David has 400 stamps more than Joyce?

10. Ali took out 4 marbles from Bag A and put into Bag B. As a result, Bag B had 5 times as many marbles as Bag A. Next, he took out 10 marbles from Bag B and put into Bag A. As a result, Bag B now had 2 times as many marbles as Bag A. How many marbles were there in Bag A at first?

1.3 DIFFERENCE UNCHANGED QUANTITIES

DEFINITION

The difference quantities remain unchanged before-change and after-change.

APPLICABILITY

There are three such situations:

- Age difference
- A third party gives the individuals involved equal amounts
- The individuals involved give away equal amounts

ILLUSTRATIONS

Age difference
A is 40 years old and B is 10 years old. Their age difference is 30 years.
4 years later, A will be 44 years old and B will be 14 years old. Their age difference is still 30 years – unchanged.

	A	B	Difference
Before	40	10	30
Change	+4	+4	
After	44	14	30

→ In the A and B change row, the change figure appears in both A and B columns.
The quantities are the same,
and the signs are also the same (+).

↓
A is older than B by 30 years.
Both A and B age by 4 years.
They will never age by different number of years.
Described another way, the difference in the change row is (4 – 4) = 0.
So, the change cell is empty (no change occurred).

Although both A and B age by 4 years,
A will still be older than B by 30 years.
Their age difference will always be 30 years.
So, the before-change and after-change Difference remain the same (Difference Unchanged Quantities).

A third party gives the individuals involved equal amounts
Amount does not just apply to money but other objects as well.
For illustration, we will use a money situation.
A has \$40 and B has \$100. The difference in their amounts is \$60.
Each of them receives \$50 from their father.
Now A has \$90 and B has \$150. The difference in their amounts is still \$60 – unchanged.

	A	B	Difference
Before	40	100	60
Change	+50	+50	
After	90	150	60

→ In the A and B change row, the change figure appears in both A and B columns.
The quantities are the same,
and the signs are also the same (+).

↓
B has \$60 more than A.

Both A and B are given an equal amount (\$50).
There is no difference in the amount they are given.
Described another way, the difference
in the change row is (50 – 50) = 0.
So, the change cell is empty (no change occurred).

Because both A and B are given an equal amount (\$50),
B will still have \$60 more than A.
So, the before-change and after-change Difference
remain the same (Difference Unchanged Quantities).

The individuals involved give away equal amounts
Similar to the above situation, except that this time the change row shows subtraction (not addition).

EXAMPLES

1. **In 2008, Keith is 5 years old and his father is 31 years old. In which year will his father's age be three times Keith's age?**

WORKING

	Keith's age	Father's age	Age difference	Year
Before	5 years	31 years	(31 – 5) years = 26 years	2008
Change	(13 - 5) years = 8 years	(39 - 31) years = 8 years		8 years
After	1 unit ↓ 1 unit x 13 = 13 years	3 units ↓ 3 units x 13 = 39 years	(3 – 1) units = 2 units ↓ 2 units x 13 = 26 years	(2008 + 8) = 2016

Keith's father will be 3 times as old as Keith in the year 2016.

EXPLANATION

List all given before-change, change and after-change information.
No conversion is needed since the information is already in units.

	Keith's age	Father's age	Age difference	Year
Before	5 years	31 years	26 years	2008
Change				
After	1 unit	3 units	(3 - 1) units = 2 units	

We know that their age difference will always be 26 years.
So, we make the age difference after-change (2 units) equal 26 years.
We do this by multiplying the age difference after-change (2 units) by 13.

Whatever we do to a number, we must also do to the other numbers in the same row to maintain the ratio.
Therefore, we must multiply the after-change row (1 unit and 3 units) by 13.

These actions will convert the after-change row's units to years.

	Keith's age	Father's age	Age difference	Year
Before	5 years	31 years	26 years	2008
Change				
After	1 unit x 13 = 13 years	3 units x 13 = 39 years	2 units x 13 = 26 years	

Now calculate and fill in the change row.

	Keith's age	Father's age	Age difference	Year
Before	5 years	31 years	26 years	2008
Change	(13 - 5) years = 8 years	(13 - 5) years = 8 years		8 years
After	1 unit x 13 = 13 years	3 units x 13 = 39 years	2 units x 13 = 26 years	

Keith's father is 3 times as old as Keith (13 - 5) **OR** (39 - 31) years,
that is 8 years later.

Keith's father is 3 times as old as Keith in the year 2016.

2. **Jasmine had 5 times as many chocolates as Rose. Both girls received 12 more chocolates from their elder sister. As a result, Rose had $\frac{1}{3}$ as many chocolates as Jasmine. How many chocolates did each girl have at first?**

WORKING

	Jasmine's chocolates	Rose's chocolates	Difference
Before	5 units	1 unit	(5 – 1) units = 4 units
Change	+ 12 chocolates ↓ (6 - 5) units = 1 unit	+ 12 chocolates ↓ (2 - 1) units = 1 unit	
After	3 parts x 2 = 6 units	1 part x 2 = 2 units	(3 – 1) parts x 2 = 4 units

1 unit → 12 chocolates
5 units → (12 x 5) chocolates = 60 chocolates

Rose had 12 chocolates, and
Jasmine had 60 chocolates at first.

EXPLANATION

List all given before-change, change and after-change information.
No conversion is needed since the information is already in units and parts.

	Jasmine's chocolates	Rose's chocolates	Difference
Before	5 units	1 unit	4 units
Change	+ 12 chocolates	+ 12 chocolates	
After	3 parts	1 part	2 parts

CONFUSION ALERT

The ratio and fraction are for different situations (before-change and after-change). That means 1 measure in the first ratio is different from 1 measure in the second fraction. Hence, we differentiate with units and parts.
And 1 unit ≠ 1 part

We know the difference in their number of chocolates remains unchanged.
So, we make the difference in their number of chocolates after-change (2 parts) equal that before-change (4 units).
We do this by multiplying the difference in their chocolates after-change (2 parts) by 2.

Whatever we do to a number, we must also do to the other numbers in the same row to maintain the ratio.
Therefore, we must multiply the after-change row (3 parts and 1 part) by 2.

These actions will convert the after-change row's parts to units.

	Jasmine's chocolates	**Rose's chocolates**	**Difference**
Before	5 units	1 unit	4 units
Change	+ 12 chocolates ↓ (6 - 5) units = 1 unit	+ 12 chocolates ↓ (2 - 1) units = 1 unit	
After	3 parts x 2 = 6 units	1 part x 2 = 2 units	2 parts x 2 = 4 units

The change in each of their chocolates is (6 - 5) **OR** (2 - 1) units, that is 1 unit.
We know that both girls received 12 chocolates from their elder sister.
1 unit → 12 chocolates

Rose had 1 unit of chocolates at first.
Jasmine had 5 units of chocolates at first.
5 units → (12 x 5) chocolates = 60 chocolates

Rose had 12 chocolates, and
Jasmine had 60 chocolates at first.

3. **Henry had \$48 and Hanna had \$20 at first. Each of them bought a similar book. As a result, Hanna was left with $\frac{1}{3}$ as much money as Henry. Find the cost of each book.**

WORKING

	Henry's money	**Hanna's money**	**Difference**
Before	\$48	\$20	\$(48 – 20) = \$28
Change	\$(48 - 42) = \$6	\$(20 - 14) = \$6	
After	3 units x 14 = \$42	1 unit x 14 = \$14	(3 – 1) units x 14 = \$28

The cost of each book was \$6.

EXPLANATION

List all given before-change, change and after-change information.
No conversion is needed since the information is already in units.

	Henry's money	Hanna's money	Difference
Before	$48	$20	$(48 – 20) = $28
Change	- $?	- $?	
After	3 units	1 unit	(3 - 1) units = 2 units

We know the difference between their money before-change and after-change remains unchanged.
So, we make the difference after-change (2 units) equal that before-change ($28).
We do this by multiplying the difference after-change (2 units) by 14.

Whatever we do to a number, we must also do to the other numbers in the same row to maintain the ratio.
Therefore, we must multiply the after-change row (3 units and 1 unit) by 14.

These actions will convert the after-change row's units to money.

	Henry's money	Hanna's money	Difference
Before	$48	$20	$28
Change	- $?	- $?	
After	3 units x 14 = $42	1 unit x 14 = $14	2 units x 14 = $28

The change in Henry's and Hanna's amount of money was $(48 - 42) **OR** $(20 - 14), that is $6.

The cost of each book is $6.

LET'S APPLY Problems Involving Difference Unchanged Quantities

1. Mr Daniel is four times as old as Dave. He is 27 years older than Dave. How old will Dave be in 7 years' time?

2. In 2009, Ali was 22 and 14 years older than Ahmad. In which year will Ahmad be $\frac{2}{9}$ as old as Ali?

3. Adrian has thrice as many stickers as Wilson. Each of them receives 15 stickers from their big brother. Now, Adrian has twice as many stickers as Wilson. How many stickers does each boy have now?

4. Kat had 168 beads more than Judy at first. After giving away the same number of beads to Natalie, Kat now has thrice as many beads as Judy. How many beads does Kat have now?

5. Lily had 12 muffins while Gillian had some muffins. Each girl bought 9 muffins from the bakery. Now Gillian has twice as many muffins as Lily. How many muffins did Gillian have at first?

6. Patrick's and Bob's savings add up to $12. Patrick has 3 times as much savings as Bob. Their father adds $5 to each of their savings. How much savings does Patrick have now?

7. Fox had 0.5 as many pencils as Terry. Each of them gave 17 pencils to their sister. Now, Terry has 3 times as many pencils as Fox. How many pencils did Terry have at first?

8. Nick collected 110 more bottle caps than Matthew. Both boys gave 16 bottle caps to their friend. Now, Matthew had $\frac{1}{6}$ times as many bottle caps as Nick. How many bottle caps did both boys have altogether at first?

9. Daniel's first English test result was $\frac{2}{3}$ as much as his first Maths test result. His first Chinese test result was 6 marks more than his first English test result. In the next class test, Daniel showed an improvement of 8 marks in each of the subjects. Given that his second Maths test result was 92 marks, how many marks did he score for the first Chinese test?

Answers to questions in the prior chapters' Let's Apply sections
are listed in this chapter. Detailed workings may be downloaded at:

www.mathsheuristics.com/solutions

CHAPTER 1 SINGLE UNCHANGED QUANTITIES

1. 144 monkeys
2. 72 apples
3. 135 pairs of earrings
4. 4 candies
5. 90 purple beads
6. 392 stamps
7. 41 marbles
8. 56 people
9. a) 80 boxes

   b) 960 pieces
10. 24 red packets

CHAPTER 2 TOTAL UNCHANGED QUANTITIES

1. 59 buttons
2. $71
3. 118 color pencils
4. 32 dolls
5. $12
6. 33 sweets
7. 52 stamps
8. 280 sweets
9. 830 stamps
10. 14 marbles

1.3 DIFFERENCE UNCHANGED QUANTITIES

1. 16 years
2. 2005
3. Adrian has 60 stickers.

   Wilson has 30 stickers.
4. 252 beads
5. 33 muffins
6. $14
7. 68 pencils
8. 186 bottle caps
9. 62 marks